Exercise 8.2.3

Show that the Chebyshev ODE, Table 7.1, may be put into self-adjoint form by multiplying by $(1-x^2)^{-1/2}$ and that this gives $w(x) = (1-x^2)^{-1/2}$ as the appropriate weighting function.

Solution

From Table 7.1 on page 345, Chebyshev's equation is

$$(1-x^2)y'' - xy' + n^2 y = 0.$$

At the moment it is not self-adjoint because

$$\frac{d}{dx}(1-x^2) \neq -x.$$

However, if both sides of Chebyshev's equation are multiplied by the weight function $w(x) = (1-x^2)^{-1/2}$, then it becomes self-adjoint

$$(1-x^2)^{1/2}y'' - x(1-x^2)^{-1/2}y' + n^2(1-x^2)^{-1/2}y = 0 \tag{1}$$

because

$$\begin{aligned}\frac{d}{dx}[(1-x^2)^{1/2}] &= \frac{1}{2}(1-x^2)^{-1/2}(-2x)\\ &= -x(1-x^2)^{-1/2}.\end{aligned}$$

Equation (1) can therefore be written as

$$(1-x^2)^{1/2}y'' + \frac{d}{dx}[(1-x^2)^{1/2}]y' + n^2(1-x^2)^{-1/2}y = 0,$$

or

$$\frac{d}{dx}[(1-x^2)^{1/2}y'] + n^2(1-x^2)^{-1/2}y = 0.$$